

Semi-classical quantization rules for a periodic orbit of hyperbolic type

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Determination of periodic orbits for a Hamiltonian system together with their semi-classical quantization has been a long standing problem. We consider here resonances for a h -Pseudo-Differential Operator $H(y, hD_y; h)$ induced by a periodic orbit of hyperbolic type at energy E_0 . We generalize the framework of [13], in the sense that we allow for both hyperbolic and elliptic eigenvalues of Poincaré map, and show that all resonances in $W = [E_0 - \varepsilon_0, E_0 + \varepsilon_0] - i]0, h^\delta]$, $0 < \delta < 1$, are given by a generalized Bohr-Sommerfeld quantization rule (BS).

1 HYPOTHESES AND THE MAIN RESULT.

Though our results apply to some manifolds, we restrict the attention to \mathbf{R}^n . Let $H(y, hD_y; h)$ be a self-adjoint h -Pseudo-Differential Operator (h -PDO) on $L^2(\mathbf{R}^n)$

$$H^w(y, hD_y; h)u(y; h) = (2\pi h)^{-n} \int \int e^{i(y-y')\eta'/h} \times H\left(\frac{y+y'}{2}, \eta'; h\right)u(y') dy' d\eta' \quad (1)$$

We assume it has Weyl symbol $H(y, \eta; h) \in S^0(m)$, where m is an order function (for example $m(y, \eta) = (1 + |\eta|^2)^M$), and

$$S^N(m) = \{H \in C^\infty(T^*\mathbf{R}^n) : \forall \alpha \in \mathbf{N}^{2n}, \exists C_\alpha > 0, |\partial_{(y,\eta)}^\alpha H(y, \eta; h)| \leq C_\alpha h^N m(y, \eta)\}$$

with the semi-classical expansion $H(y, \eta; h) \sim H_0(y, \eta) + hH_1(y, \eta) + \dots$, $h \rightarrow 0$. Here H_0 is the principal symbol of H , H_1 its sub-principal symbol. We assume that $H(y, \eta; h) + i$ is elliptic, i.e. $|H(y, \eta; h) + i| \geq \text{const.} m(y, \eta)$, and extends as an analytic symbol in the sense of [25] in a sector

$$\Gamma_0 = \{(y, \eta) \in T^*\mathbf{C}^n : |\text{Im}(y, \eta)| \leq \text{const.} \langle \text{Re}(y, \eta) \rangle\}$$

Let the energy surface $H_0^{-1}(E_0)$ be regular for some $E_0 \in \mathbf{R}$, that we may set up to 0. So the Hamiltonian vector field X_{H_0} has no fixed point on $H_0^{-1}(0)$,

hence on nearby energy surfaces $H_0^{-1}(E)$. Let $\Phi^t = \exp(tX_{H_0}) : T^*\mathbf{R}^n \rightarrow T^*\mathbf{R}^n$ and

$$K_E = \{\rho \in H_0^{-1}(E), \Phi^t(\rho) \not\rightarrow \infty \text{ as } |t| \rightarrow \infty\} \quad (2)$$

be the trapped set at energy E . We assume that $K_0 = \gamma_0$ is a periodic orbit of period T_0 , or possibly a finite union of such orbits. Let \mathcal{P}_0 be Poincaré map, or the first return map, acting on Poincaré sections $\Sigma(\rho) \subset T^*\mathbf{R}^n$, $\rho \in \gamma_0$. Assume also that γ_0 is non degenerate, i.e. 1 is not an eigenvalue of $d\mathcal{P}_0$, then it follows from Poincaré Continuation Theorem that for $|E|$ small enough, γ_0 belongs to a one parameter family (the center manifold) of such periodic orbits $K_E = \gamma_E$ of period T_E . We identify the center manifold with a neighborhood of the zero-section in $T^*\mathbf{S}^1$, and each $\Sigma(\rho)$, $\rho \in \gamma_0$ with $\Sigma \approx T^*\mathbf{R}^d$, modulo the action of Hamiltonian flow. Both $\overline{\gamma}$ and Σ are symplectic manifolds. For $\rho \in \gamma_0$, let λ_j , $1 \leq j \leq 2d = 2(n-1)$ be the eigenvalues of $A_0 = d\mathcal{P}_0 : \mathbf{C}^{2d} \rightarrow \mathbf{C}^{2d}$ (Floquet multipliers).

The space \mathbf{C}^{2d} has the orthogonal symplectic decomposition in (generalized) eigenspaces F_λ relative to the family $(\lambda_j)_{1 \leq j \leq 2d}$. We are interested in the case where A_0 is partially hyperbolic, i.e. has at least one eigenvalue λ of modulus $\neq 1$. Assume also that Poincaré map is non degenerate, i.e. $F_1 = \{0\}$, and also $F_\lambda = \{0\}$ for all $\lambda \leq 0$. We say that $\lambda \in \mathbf{C}$ is *elliptic* (ee for short) if $|\lambda| = 1$ ($\lambda \neq \pm 1$) and *hyperbolic* (he) if $|\lambda| \neq 1$; if moreover $\lambda \in \mathbf{R}$ we call it *real hyperbolic* (hr) and *complex-hyperbolic* (hc) otherwise. Under the last assumption we can define $B_0 = \log A_0$. Eigenvalues $\mu = \mu(\lambda) = \log \lambda$ of B_0 (Floquet exponents) verify $\mu(\overline{\lambda}) = \overline{\mu(\lambda)}$. Accordingly, exponent μ is said ee if $\text{Re } \mu = 0$, hr if $\mu \in \mathbf{R} \setminus 0$, and hc if $\mu \in \mathbf{C} \setminus \mathbf{R}$. So eigenvalues of B_0 have the form $\mu_j, -\mu_j$ in the hr sector, $\mu_j, \overline{\mu_j}$ in the ee sector, and $\mu_j, -\mu_j, \overline{\mu_j}, -\overline{\mu_j} \neq 0$, $\text{Re } \mu_j > 0$, (with same multiplicity) in the hc sector. Every pair of simple Floquet exponents $\mu_j, -\mu_j$ in the ee sector can be ordered thanks to the symplectic

structure on \mathbf{C}^{2d} . Namely if u is an eigenvector corresponding to such a μ , then $\frac{1}{2i}\sigma(\bar{u}, u) \in \mathbf{R} \setminus 0$ with a sign independent of u ; so we call μ an *eigenvalue of the first kind* if $\frac{1}{2i}\sigma(\bar{u}, u) > 0$. Alternatively, the Hamiltonian flow of $b(\rho) = \frac{1}{2}\sigma(\rho, B_0\rho)$ (Hermitian form) with time of negative imaginary part is “expanding” on an eigenvector corresponding to an eigenvalue of the first kind. Let r be the number of distinct μ_j ’s. For simplicity, assume $r = d$, hence b diagonalizable. We know [4] that in a suitable basis $b(\rho)$ is a linear combinaison of elementary quadratic polynomials Q_j , and that we can choose complex symplectic coordinates, so that Q_j takes the form $Q_j = x_j \xi_j$. The Q_j ’s play an important rôle, since they are formally “transverse eigenvectors” for H , microlocalized near γ_0 .

Our next Hypothesis is relative to *partial hyperbolicity* of Poincaré map, in the sense that there exists $j \in \{1, \dots, r\}$, such that $\operatorname{Re} \mu_j > 0$. For hyperbolic dynamical systems, we know [1] that there are generically elliptic elements. Let F_{μ_j} , $\operatorname{Re} \mu_j \geq 0$ denote again the eigenspace associated with μ_j . We can rewrite the decomposition of \mathbf{C}^{2d} in the sum of unstable $F^+ = \bigoplus_{j=1}^{r=d} F_{\mu_j}$, resp. stable $F^- = \bigoplus_{j=1}^{r=d} F_{-\mu_j}$ spaces where $F^\pm \simeq \mathbf{C}^d$ are (complex) Lagrangian subspaces of \mathbf{C}^{2d} , and F^+ contains the eigenspaces associated with eigenvalues of the first kind.

Our last Hypothesis concerns the non-resonance condition relative to Floquet exponents, which is required to achieve Birkhoff normal form, namely

$$\forall k \in \mathbf{Z}^d : \sum_{j=1}^d k_j \mu_j \in 2i\pi\mathbf{Z} \implies \sum_{j=1}^d k_j \mu_j = 0 \quad (3)$$

For instance, when $n = 2$ and $\mu_1 = i\omega_1$, it means that the rotation number ω_1 is irrational. We need also the strong non-resonance condition on Floquet exponents:

$$\forall k \in \mathbf{Z}^d : \sum_{j=1}^d k_j \mu_j \in 2i\pi\mathbf{Z} \implies k = 0 \quad (4)$$

Examples: 1) The Model Hamiltonian

$$\tilde{H}(hD_t, x, hD_x; h) = -hD_t + \sum_{j=1}^d \mu_j Q_j^w(x, hD_x) \quad (5)$$

$Q_j^w(x, hD_x) = \frac{1}{2}(x_j hD_{x_j} + hD_{x_j} x_j)$, with Periodic Boundary Conditions on $\mathbf{S}^1 \times \mathbf{R}^d$, serves as a guideline throughout this work. Here x may denote com-

plex variables, in some Bargmann representation of the Hamiltonian.

2) A physical example is given by $H(y, hD_y) = -h^2 \Delta_y + |y|^{-1} + ay_1$ on \mathbf{R}^n (repulsive Coulomb potential perturbed by Stark effect) near an energy level $E > 2/\sqrt{a}$. More generally, Schrödinger operators with potentials with two or more bumps, which are semi-classical analogues of Helmholtz operators with obstacles, are considered in [27]. Such a periodic orbit is sometimes called a *libration* in the case of a compact energy surface.

3) The geodesic flow on the one-sheeted hyperboloid in \mathbf{R}^3 has an (unstable) periodic orbit of hyperbolic type (Poincaré example). This example generalizes ([6], App.C) to a surface of revolution in \mathbf{R}^4 , involving two symmetric periodic orbits of real-hyperbolic type at energy E_1 , and a periodic orbit of mixed type in between, at energy $E_2 < E_1$ (with real-hyperbolic and elliptic elements). Our method carries to the case when $H(y, hD_y)$ is the geodesic flow on such manifolds, and K_E the union of two periodic orbits, provided we modify Grushin operator of [28] accordingly.

We are concerned with semi-classical *resonances* of H near 0, in the framework of “complex scaling” theory and its extensions [23], [11], i.e. the discrete spectrum in the lower-half plane of some suitable analytic continuation of H as a closed, Fredholm, but non-selfadjoint operator. Our main result, the generalized Bohr-Sommerfeld quantization condition, can be formulated as follows (see also [19]):

Theorem 1 *Under the hypotheses above, let (after re-ordering) $\mu_j = i\omega_j$, $j = 1, \dots, \ell$, $\omega_j > 0$ be the set of elliptic Floquet exponents for H_0 . Recall H_1 from (??), and let $H_1(y(t), \eta(t)) dt$ the sub-principal 1-form. We define the semi-classical action along γ_E , by $\mathcal{S}(E; h) = S_0(E) + hS_1(E) + \mathcal{O}(h^2)$ with*

$$S_0(E) = \frac{1}{2\pi} \int_{\gamma_E} \xi dx \quad (6)$$

$$S_1(E) = -\frac{1}{2\pi} \int_0^{T(E)} H_1(y(t), \eta(t)) dt + \quad (7)$$

$$\frac{1}{4i\pi} \sum_{j=1}^d \mu_j(E) + \frac{g_\ell}{4}$$

Here $\mu_j(E) = \mu_j + \mathcal{O}(E)$ is Floquet exponent at energy E , $g_\ell \in \mathbf{Z}$ Gelfand-Lidskii (or Cohnley-Zehnder) index of γ_E (depending only on elliptic elements, see below). Then given any $0 < \delta < 1$, the resonances of H near 0 in the “window” $W_h =$

$[-\varepsilon_0, \varepsilon_0] - i]0, h^\delta]$ ($\varepsilon_0 > 0$ sufficiently small) are given (at first order in h) by the generalized BS quantization condition

$$\mathcal{S}(E; h) + \frac{1}{2i\pi} \left(\sum_{j=1}^d k_j \mu_j(E) + \mathcal{O}(h|k|^2) \right) = mh \quad (8)$$

with $m \in \mathbf{Z}$, $k \in \mathbf{Z}^d$, provided $|m|h \leq \varepsilon_0$, $|k|h \leq \text{const. } h^\delta$.

In the elliptic case, a similar theorem (for real spectrum) was obtained in [2], [3], and [22]; in the real hyperbolic case, in [13], [14] for $|\text{Im } E| = \mathcal{O}(h)$, and [26] in dimension 2 with $|\text{Im } E| = \mathcal{O}(h^\delta)$ or even larger, but selecting a single Floquet parameter in the semi-classical Floquet decomposition of H near γ_0 , i.e. few “longitudinal” or “principal” quantum numbers $m \in \mathbf{Z}$. For related results about trace formulas or concentration of eigenvalues in the compact case, see [29], [30], [28], [21], [6]. For the wave equation outside convex obstacles, where all Floquet exponents of the billiard map are real-hyperbolic, see [18], [12].

It is well known [28], [21], [9] that all resonances in W_h are given by the zeroes of a “ ζ -function” $\zeta(z; h) = \det(\text{Id} - N(z; h))$, where $N(z; h) = \Pi_N(h)M(z; h)^*\Pi_N(h) + \mathcal{O}(h^N)$ is the approximation of order N of a semi-classical *monodromy operator* $M(z; h)^*$ in a suitable (diagonal) basis of homogeneous polynomials. We compute here the semi-classical action at first order, which is the main contribution of $\zeta(z; h)$. We make an essential use of Birkhoff normal form in the neighborhood of γ_0 . Higher approximations could be obtained as in the 1-D case [17].

2 CONSTRUCTION OF THE MONODROMY OPERATOR

The main object to be constructed is the monodromy operator $M^*(E; h)$, a h -FIO defined on a Poincaré section and quantizing Floquet operator associated with the periodic orbit.

1) Birkhoff normal form (BNF).

We start to find suitable coordinates near $\bar{\gamma}$. When $\text{Re } \mu_j > 0$ for all $1 \leq j \leq d$, the stable/unstable manifold theorem guarantees the existence of involutive manifolds Γ_\pm in a neighborhood of γ_0 with $T_\rho \Gamma_+ + T_\rho \Gamma_- + T_\rho \bar{\gamma} = T^*\mathbf{R}^n$, $\rho \in \gamma_0$. There are (real) symplectic coordinates (t, τ, x, ξ)

such that $\xi = 0$, $d\xi \neq 0$ on Γ_+ , $x = 0$, $dx \neq 0$ on Γ_- , and (t, τ) parametrize $\bar{\gamma}$. Intersecting with the energy surfaces $H_0^{-1}(E)$ gives the splitting $T_\rho \Gamma_+(E) + T_\rho \Gamma_-(E) + T_\rho \gamma(E) = T_\rho^* H_0^{-1}(E)$, $\rho \in \gamma(E)$, and $\Gamma_\pm(E)$ are Lagrangian submanifolds in $T^*\mathbf{R}^n$ intersecting transversally along $\gamma(E)$. In these coordinates $H_0(y, \eta) = f(\tau) + \langle B(t, \tau, x, \xi)x, \xi \rangle$. Here f parametrizes the energy parameter $f(\tau) = E$, and is related with the period $T(E)$ of $\gamma(E)$ by $f'(\tau) = \frac{2\pi}{T \circ f(\tau)}$. Performing a first canonical transformation gives $B(t, \tau, x, \xi) = \tilde{B}_0 + \mathcal{O}(|\tau|, |x, \xi|)$, where the eigenvalues of \tilde{B}_0 are Floquet exponents for $d\mathcal{P}_0$ with positive real part. When μ_k is an eigenvalue of the first kind, splitting of $T_\rho^* H_0^{-1}(E)$ as above still holds provided we take complex variables. In both cases however, under the non resonance conditions (??), (??) BNF holds in the classical sense [5] as well as in the semi-classical sense [15] and takes, modulo a small remainder term, operator $H^w(y, hD_y; h)$ to a polynomial in hD_t and $Q_j^w(x, hD_x)$, $Q_j(x, \xi)$ as above. In particular, the principal part of $H^w(y, hD_y; h)$ is in BNF is given by (??) in some suitable Bargmann (still formal) representation, provided a reparametrization of energy.

2) Microlocalisation in the complex domain.

Resonances here are considered from the point of view of analytic dilations and Lagrangian deformations; taking into account that there exists an *escape function* (that grows along the flow of X_{H_0}) which implies kind of a “virial condition” outside the trapped set γ_0 , the most relevant region of phase-space for such deformations is a neighborhood of γ_0 . Here we make a complex scaling of the form $(x, \xi) \mapsto (e^{i\theta}x, e^{-i\theta}\xi)$, followed also by a small deformation in the (t, τ) variables. Our main tool is FBI transformation (metaplectic FIO with complex phase, see [25], [20]) which takes the form, in coordinates $(s, y; t, x) \in T^*\mathbf{R}^n \times T^*\mathbf{C}^n$ adapted to Γ_\pm as in BNF, and acts on $u \in L^2(\mathbf{R}^n)$ as

$$T_0 u(x, h) = \int e^{i\varphi_0(t, s; x, y)/h} u(s, y) ds dy$$

where $\varphi_0(t, s; x, y) = \varphi_1(t, s) + \varphi_2(x, y)$, $\varphi_1(t, s) = \frac{i}{2}(t-s)^2$, $\varphi_2(x, y) = \frac{i}{2}[(x-y)^2 - \frac{1}{2}x^2]$. The corresponding canonical transformation is $\kappa_0 = (\kappa_1, \kappa_2)$, with $\kappa_1 : (s, -\partial_s \varphi_1) \mapsto (t, \partial_t \varphi_1)$, $\kappa_2 : (y, -\partial_y \varphi_2) \mapsto (x, \partial_x \varphi_2)$, and the corresponding pluri-subharmonic (pl.s.h.) weight $\Phi_0 = \Phi_1 + \Phi_2$,

with $\Phi_1(t) = (\text{Im } t)^2/2$, $\Phi_2(x) = |x|^2/4$. In a very small neighborhood of γ_0 , whose size will eventually depend on h , corresponding to $\theta = -\pi/4$, and that we call the “phase of inflation”, it turns out that $H^w(y, hD_y; h)$ takes the simple form above whose principal term is given in (??), and the corresponding weight $\tilde{\Phi}(t, x)$ is just $\Phi_0(t, x)$. Otherwise we take θ small enough in a somewhat larger neighborhood of γ_0 , which we call the “linear phase”. Farther away from γ_0 (in the “geometric phase”) the weight is implied by the escape function given by the non-trapping condition. All these weights are patched together in overlapping regions, so to define a globally pl.s.h. function in \mathbf{C}^n . They also define the contour integral for writing a h -FIO in the complex domain, using almost analytic extensions because of the loss of analyticity in BNF. After these transformations, $H_0 - E$ becomes elliptic everywhere but on γ_E .

3) Poisson operator and its normalisation.

Let \mathbf{R}_t^n be the section $\{t\} \times \mathbf{R}^d$ of \mathbf{R}^n (in BNF coordinates). We look for $K(t, E) : L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}_t^n)$ (formally), microlocalized near $\Gamma_+(E)$, such that $H(hD_t, x, hD_x; h)K(t, E) = 0$, $K(0, E) = \text{Id}$. In the “phase of inflation”, considering realizations in the complex domain adapted to the weight $\tilde{\Phi}_\theta$, it takes the form $K(t, E)v(x; h) = \int \int e^{i(S(t, x, \eta) - y\eta)/h} a(t, x, \eta; E, h)v(y) dy \wedge d\eta$. Solving eikonal and transport equations, we find that the leading term of S and a with respect to BNF is given by those of (??), and $K(t, E)$ is also in BNF. Let $\chi \in C^\infty(\mathbf{R})$, be equal to 0 near 0, 1 near $[2\pi, \infty[$. There is a h -PDO $B(E) = B^w(x, hD_x; E)$ such that $L(t, E) = K(t, E)B(E)$ satisfies the “flux norm” identity of [28]

$$\left(\frac{i}{h}[H, \chi(t)]L(t, E)v|L(t, E)v\right) = (v|v) \quad (9)$$

Considering also microlocal reproducing kernels, we show there exists $P(s, E) = \text{Id}_{L^2(\mathbf{R}_s^n)} + \mathcal{O}(h)$ such that

$$\begin{aligned} \int dE L(\cdot, E)P(\cdot, E)L^*(E) &= \text{Id}_{L^2(\mathbf{R}^n)} \\ \int dE L^*(E)L(s, E)P(s, E) &= \text{Id}_{L^2(\mathbf{R}_s^n)} \end{aligned} \quad (10)$$

These (formal) computations can be carried out in the framework of FIO’s in the complex domain.

4) The monodromy operator.

We set $K_0(t, E) = K(t, E)$ where $K(t, E)$ is Poisson operator with Cauchy data at $t = 0$, and $L_0(t, E) = K_0(t, E)B(E)$; we set similarly $L_{2\pi}(t, E) = K_0(t - 2\pi, E)B(E)$ with Cauchy data at $t = 2\pi$. The monodromy operator (or semi-classical Poincaré map) is defined by

$$M^*(E) = L_{2\pi}^*(E) \frac{i}{h} [H, \chi] L_0(\cdot, E) \quad (11)$$

as an operator on $L^2(\mathbf{R}^d)$. As a function of χ , $M^*(E)$ follows a “0-1 law”: it is 0 if $\text{supp } \chi \subset]0, 2\pi[$, and unitary if χ equals 0 near 0, and 1 near 2π . Unitarity is shown using (??) and (??). For the model case one has $M^*(E)v(x) = e^{-2i\pi E/h} e^{\pi i \mu} v(xe^{2\pi i \mu})$ when $\int \chi'(t) dt = 1$. Moreover Schwartz kernel of $M^*(E)$ given by

$$\mathcal{M}^*(t, x, z) = \chi'(t) \int e^{i\Psi_2(t, x, z, \eta')/h} m^* d\eta' \quad (12)$$

(where $m^* = m^*(t, x, z, \eta'; h)$ is a suitable symbol) is in BNF, and the integral is independent of t . In fact $M^*(E) = e^{iR^w(x, hD_x; E, h)/h}$, where R is h -PDO in BNF, self-adjoint for real E .

3 THE QUANTIZATION CONDITION

Reducing the spectral problem for $H^w(x, hD_x; h)$ through a Grushin operator as in [28], [21], [9], we consider the approximate kernel of $M^*(E) - \text{Id}_{L^2(\mathbf{R}^d)}$, after taking suitable analytic extensions with respect to E of Poisson and monodromy operators. The discrete set of E ’s such that 1 belongs to the spectrum of $M^*(E)$ is precisely the set of resonances. We follow [13], [22], while taking further advantage of BNF to compute resonances lying farther from the real axis. Since $M^*(E)$ is in BNF (formal) eigenfunctions of $M^*(E)$ are given by homogeneous polynomials $f_k(x)$, whose degree $|k|$ depends on the accuracy of BNF; we can take $|k| = h^{-\delta}$, for some $0 < \delta < 1$. So we need to compute $M^*(E)f_k$ by asymptotic stationary phase using (??), the leading term being $f_k(x)$ times the phase factor $e^{iS(E; h)/h}$ involving the semi-classical action $S(E; h) = S_0(E) + hS_1(E) + \mathcal{O}(h^2)$, and times a half-density. Still we shall formally discuss the various terms in (??)-(??), without resorting to $M^*(E)$.

1) Action integral and half-density.

In (??) we can readily identify the classical action $S_0(E) = \frac{1}{2\pi} \int_{\gamma_E} \xi dx$, as well as the principal part of a half-density. These are the “constant terms” (independent of the transverse variables) in $M^*(E)f_k$ (the kernel of $M^*(E)$ being given in (??)), which contribute to the “longitudinal quantum number” m as in the 1-D case. Namely, looking for a WKB solution, we solve

$$(H^w - E)(a_{\pm}^E(y; E, h) \exp[i\phi_{\pm}^E(y, E)/h]) = 0$$

and only take in account the “outgoing” solution (+). The solution to the eikonal equation $H_0(y, \frac{\partial \phi^E}{\partial y}) = E$, is a multivalued function ϕ^E , such that if we denote by \mathcal{J}_E the “first return” map (in space variable), then $\phi^E(\mathcal{J}_E y) = \phi^E(y) + C(E)$, where $C(E) = 2\pi S_0(E)$ verifies $S'_0(E) = T(E)$. Solving next for the first transport equation, we find accordingly $a_0^E(\mathcal{J}_E y) = \rho(E)a_0^E(y)$, where $\rho(E)$ can be computed in the coordinates of Sect.4.1. Assume for simplicity that the subprincipal symbol $H_1(y, \eta) = 0$. If $d\mathcal{P}_E = \begin{pmatrix} A(E) & 0 \\ 0 & {}^t A(E)^{-1} \end{pmatrix}$ denotes linearized Poincaré map, we have $|\rho(E)| = |\det A(E)|^{-1/2} = |\prod_{j=1}^d \lambda_j(E)|^{-1/2}$, where $\lambda_j(E)$ belong to the set of eigenvalues of $d\mathcal{P}_E$ of the first kind (of modulus 1), or of modulus > 1 , see [13], Prop.1.1. Thus in $\mathcal{S}(E; h)$ and (??) we have identified $S_0(E)$ and the term $\frac{1}{4i\pi} \sum_{j=1}^d \mu_j(E)$ of $S_1(E)$. We can also identify the contribution of polynomials f_{α} in $M^*(E)f_{\alpha}$. Namely, let D^E be the restriction of $d\mathcal{P}_E$ to F_+ , and $D_*^E : \mathcal{Q}^N(F_+) \rightarrow \mathcal{Q}^N(F_+)$ its push-forward acting on the space of polynomials of degree $\leq N$. The eigenvalues of D_*^E are then the numbers $\lambda(E)^{-k} = \lambda_1(E)^{-k_1} \dots \lambda_d(E)^{-k_d}$, with $|k| = k_1 + \dots + k_d$. This gives the “transverse” quantum numbers k_j and the sum on the LHS of (??), the remainder $\mathcal{O}(|k|^2)$ being given by higher order terms in BNF.

2) Cohnley-Zehnder index.

We are left with the RHS of (??). We recall [10], [22], [7], [24] the index of a symplectic arc (Gelfand-Lidskii, or Conley-Zehnder), which appears in the quantization condition when elliptic elements occur. It is defined for a differentiable path $\Psi : [0, T] \rightarrow \text{Sp}(2n; \mathbf{R})$ such that $\Psi(0) = \text{Id}$ and $\det(\text{Id} - \Psi(T')) \neq 0$ for some $T' \in [0, T]$. In the present case, let $Z(s)$ solve the variational system

along γ_E , i.e. $\dot{Z}(s) = JH''(\Phi^s(\rho))Z(s)$, $Z(0) = \text{Id}$, where we recall $\Phi^s(\rho)$ is the flow of X_{H_0} issued from $\rho \in \gamma_E$ as in (??). We define $\Psi(s) = d\mathcal{P}_E(s)$ as the co-restriction $\tilde{Z}(s)$ of $Z(s)$ to Poincaré sections, i.e. $\Psi : [0, T] \rightarrow \text{Sp}(2d; \mathbf{R})$. Then, Conley-Zehnder index can be interpreted as the mean winding number of the eigenvalues of the first kind, or equivalently the number of such eigenvalues crossing 1. Here we take advantage of BNF and the splitting of $T^*\mathbf{R}^d$ in the incoming/outgoing manifolds to simplify the argument of [24].

Recall $\Lambda \in \mathcal{L}(2d)$ (the Grassmanian of Lagrangian planes), can be represented as $\Lambda = \mathcal{U}(\mathbf{R}^d)$, $\mathcal{U} \in U(d)$ (unitary group); given a differentiable path $\mathbf{S}^1 \rightarrow \mathcal{L}(2d)$, $s \mapsto \Lambda_s = \mathcal{U}_s(\mathbf{R}^d)$, Maslov universal class is defined by

$$dm(\Lambda_s) = dm_s = \frac{1}{4\pi} d(\arg \det \mathcal{U}_s^2) \quad (13)$$

Consider for simplicity the linear situation, i.e. Hamiltonian (??), without loss of generality we may assume $T = T' = 1$. Integrating the variational system we find $\tilde{Z}(s) = \begin{pmatrix} C(s) & B(s) \\ B(-s) & C(-s) \end{pmatrix}$, where $\begin{pmatrix} C(s) \\ B(s) \end{pmatrix}$, $C(s), B(s)$ symmetric, is a basis of the Lagrangian subspace $\Lambda_+(s) = \exp sX_H(\Gamma_+(0))$, and $\begin{pmatrix} B(-s) \\ C(-s) \end{pmatrix}$ (by Fourier transform) a basis of $\Lambda_-(s) = \exp sX_H(\Gamma_-(0))$. As in [8] we introduce Cayley transformation

$$\mathcal{U}(s) = (C(s) + iB(s))(C(s) - iB(s))^{-1}$$

which parametrizes $\Lambda_+(s)$ and $\mathcal{U}(s)$ is unitary except for finitely many points $s \in [0, 1]$, corresponding to discontinuities of dm_s at the passage of a caustics (i.e. whenever $s\omega_k \in 2\pi\mathbf{Z}$, for some s, k , $\mu_k = i\omega_k$ being a Floquet exponent of the first kind). Let $d\tilde{m}_s$ be the regular part of dm_s , by a direct computation $\frac{1}{4\pi} \int_0^{2\pi} d\tilde{m}(\Lambda(s)) = 2 \sum_{k=1}^{\ell} \omega_k$, where the summation runs over elliptic elements. The variation of dm_s at a caustics can be determined as in ([16], Sect.3.3), using also the non degeneracy of Poincaré map. This gives (??) in the model case. The general case follows from BNF and stability of index g_{ℓ} under perturbation. Note that computing $M^*(E)f_k$ by stationary phase readily gives $g_{\ell} \bmod 4$.

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